Lie Symmetry Analysis, Optimal System and Exact Solutions of the (2+1)-Dimensional Negative-Order Modified Calogero–Bogoyavlenskii–Schiff (nmCBS) Equation

A DISSERTATION

Submitted in partial fulfillment of the

requirements for the award of the degree of

Master of Science

in

Mathematics

Submitted by

PRIYANSHU YADAV

2023PMA6932

Under the supervision of

Dr. Deepika Singh

Department of Mathematics

NETAJI SUBHAS UNIVERSITY OF TECHNOLOGY

(Formerly Netaji Subhas Institute of Technology)

AZAD HIND FAUZ MARG, SECTOR-3, DWARKA, NEW DELHI-110078

May, 2025

Department of Mathematics
NETAJI SUBHAS UNIVERSITY OF TECHNOLOGY
(Formerly Netaji Subhas Institute of Technology)
AZAD HIND FAUZ MARG, SECTOR-3, DWARKA, NEW DELHI-110078

CANDIDATE'S DECLARATION

I, **PRIYANSHU YADAV**, Roll No. **2023PMA6932** student of **M.SC. MATHEMATICS**, hereby declare that the project Dissertation titled **"Lie Symmetry Analysis, Optimal System and Exact Solutions of the (2+1)-Dimensional Negative-Order Modified Calogero–Bogoyavlenskii–Schiff (nmCBS) Equation"** which is submitted by me to the **Department of Mathematics**, **Netaji Subhas University of Technology, Delhi** in partial fulfillment of the requirement for the award of the degree of Master of Science is original and not copied from any source without proper citation.
This work has not previously formed the basis for the award of any Degree, Diploma Associateship, Fellowship, or other similar title or recognition.

Place: Delhi
Date: May 15, 2025

PRIYANSHU YADAV

Department of Mathematics
NETAJI SUBHAS UNIVERSITY OF TECHNOLOGY
(Formerly Netaji Subhas Institute of Technology)
AZAD HIND FAUZ MARG, SECTOR-3, DWARKA, NEW DELHI-110078

CERTIFICATE

I hereby certify that the project Dissertation titled **"Lie Symmetry Analysis, Optimal System and Exact Solutions of the (2+1)-Dimensional Negative-Order Modified Calogero–Bogoyavlenskii–Schiff (nmCBS) Equation"** which is submitted by **PRIYANSHU YADAV**, Roll No. **2023PMA6932**, **M.SC. MATHEMATICS, Netaji Subhas University of Technology, Delhi** in partial fulfillment of the requirement for the award of the degree of Master of Science, is a record of the project work carried out by the student under my supervision. To the best of my knowledge, this work has not been submitted in part or full for any Degree or Diploma to this University or elsewhere.

Dr. Deepika Singh
SUPERVISOR

Place: Delhi
Date: May 15, 2025

ACKNOWLEDGEMENT

With deepest appreciation, I extend a special thank you to my mentor, Dr. Deepika Singh, Assistant Professor, Department of Mathematics, Netaji Subhas University of Technology, Delhi, for her unwavering guidance and inspiration throughout my project, “Lie Symmetry Analysis, Optimal System and Exact Solutions of the (2+1)-Dimensional Negative-Order Modified Calogero–Bogoyavlenskii–Schiff (nmCBS) Equation”.

Her mentorship made this journey truly rewarding. I also acquired valuable skills that I look forward to leveraging in the future. My heartfelt thanks go to my colleagues and friends for their constant support and insightful suggestions.

PRIYANSHU YADAV

Contents

LIST OF FIGURES

LIST OF TABLE

ABSTRACT

The aim of this thesis is to investigate the non-linear (2+1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff (nmCBS) equation, derived from the truncated Painlevé expansion, using the Lie group theoretic approach. We have thoroughly explored the characteristics of non-linear partial differential equations (PDEs), their applications in modeling complex physical systems, and the Lie group theoretic approach. For a deeper understanding, foundational concepts from the prior work done on Lie symmetry method have been revisited. In this study, we apply the Lie group theory to analyze the nmCBS equation, which poses significant challenges for analytical solutions due to its non-linearity. The Lie group theoretic approach is used to find the similarity reductions and analytic solutions of the (2+1)-dimensional nmCBS system. The infinitesimal generators for the considered system are obtained under the invariance property of the Lie group of transformations. Later, we construct groups of symmetries and tables for commutation and adjoints. The adjoint table is further used to establish a one-dimensional optimal system of subalgebras. Finally, based on the optimal system, similarity reductions are obtained. A repeated process of similarity reductions reduces the governing system of partial differential equations (PDEs) into systems of ordinary differential equations (ODEs) that generate invariant solutions. Moreover, the dynamical behaviors of the obtained solutions such as kink solution, constant solution, singular solution and multi-soliton waves are graphically shown using 3D, 2D, and corresponding contour plots. These solutions capture dispersion, non-linearity, and coupling effects, consistent with existing literature.

Keywords: (2+1)-dimensional nmCBS equation, Lie symmetry method, Optimal solution, Exact solution, Invariant solution, Kink, Soliton, Constant solution.

Chapter 1

INTRODUCTION

This chapter introduces the study of non-linear partial differential equations (PDEs) and their analysis using the Lie symmetry method, focusing on the non-linear (2+1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff (nmCBS) system derived from the truncated Painlevé expansion. It covers fundamental concepts, the importance of non-linear PDEs, the role of symmetry in their solution, and the Lie symmetry method. The purpose is to provide a foundation for investigating the (2+1)-dimensional nmCBS system of PDEs through Lie symmetry analysis, offering motivation and an overview of the topic.

1.1 Motivation and Overview

Non-linear partial differential equations are a vital link between mathematics and science, playing a key role in fields like physics, engineering, and biology. Unlike linear PDEs, non-linear PDEs exhibit complex behaviors such as solitons, shock waves, and blow-up phenomena, making exact solutions difficult to obtain.

These equations model dynamic systems where quantities like velocity, pressure, or density vary over space and time, as seen in fluid dynamics, non-linear optics, and wave propagation. While numerical methods are common, exact solutions, when possible, offer valuable insights and serve as benchmarks for computational approaches. The nmCBS equation, a non-linear PDE, is of interest due to its relevance in modeling complex wave interactions and dispersive phenomena. Solving such equations often requires reducing their complexity, typically by exploiting symmetries.

The Lie symmetry method, which identifies transformations that preserve a PDE's structure, is an effective approach. By finding similarity variables, this method reduces the number of independent variables, transforming PDEs into simpler ordinary differential equations (ODEs). This thesis applies Lie symmetry analysis to the (2+1)-dimensional nmCBS system of PDEs to derive exact solutions, such as traveling waves, that capture its non-linear and dispersive characteristics.

1.2 Role of Non-linear PDEs in Science

Non-linear PDEs are essential for describing complex physical phenomena. In physics, the Korteweg-de Vries (KdV) equation models solitons in shallow water waves, while the non-linear Schrödinger equation governs solitary waves in optics and quantum mechanics. In fluid dynamics, the Burgers' equation describes shock wave formation, relevant to traffic flow and gas dynamics. Biology uses reaction-diffusion equations to model pattern formation, such as in embryonic development, while epidemiology employs PDEs to track disease spread. In finance, non-linear extensions of the Black-Scholes equation address complex derivatives. These examples highlight the versatility of non-linear PDEs in translating physical laws into mathematical frameworks, despite their analytical challenges due to phenomena like wave steepening and discontinuities.

1.3 The Concept of Symmetry in Mathematics

Symmetry is a central concept in mathematics with applications in geometry and algebra. Geometrically, symmetry involves transformations—rotations, reflections, or translations—that preserve an object's properties. Algebraically, symmetries form groups, satisfying closure, associativity, identity and invertibility.

Sophus Lie extended this to continuous symmetries through Lie groups, smooth manifolds with group structures (e.g., the rotation group $SO(2)$), and Lie algebras, which describe infinitesimal transformations. For non-linear PDEs, Lie symmetry analysis identifies transformations that leave the equation invariant, enabling dimensionality reduction and group invariant solutions. By solving determining equations, we derive infinitesimal generators. These, along with vector fields, help identify optimal subalgebras. Using invariant conditions, these subalgebras transform the PDE into a solvable ODE.
Noether's theorem links symmetries to conservation laws, such as energy or momentum conservation, enhancing physical interpretations. This thesis builds on these concepts to analyze the (2+1)-dimensional nmCBS system, aiming to derive exact solutions.

1.4 Some Basic Definitions

1.4.1 Lie Group

A Lie group is a fundamental tool in symmetry analysis, widely employed to derive optimal subalgebras for solving problems in gas dynamics and wave propagation, particularly those involving shock phenomena. It consists of a set of transformations defined as

$$\bar{x}_i = f_i(x_j, \varepsilon), \quad i, j = 1, 2, \ldots, \tag{1.4.1.1}$$

where ε is a parameter. These transformations form a one-parameter Lie group if they satisfy the axioms of closure, associativity, inverse, and identity with respect to a composition law $\phi(a, b)$. Additionally, a Lie group is a smooth manifold, a geometric structure that facilitates the study of continuous symmetries in differential equations.

1.4.2 Optimal Subalgebra

An optimal subalgebra refers to a systematically classified set of symmetry subalgebras derived from a Lie group, enabling the reduction of a partial differential equation to simpler forms while preserving its solution structure. This approach ensures that all non-equivalent solutions are captured through symmetry reductions. For example, consider the transport equation

$$u_t + cu_x = 0, \tag{1.4.2.1}$$

where $u(x,t)$ is a wave function and c is the wave speed. To derive solutions using an optimal subalgebra, introduce dimensionless variables

$$\xi = x - ct, \quad \eta = t. \tag{1.4.2.2}$$

Define

$$v(\xi,\eta) = u(x,t), \quad u_x = \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{\partial v}{\partial \xi}, \quad u_t = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial t} = \frac{\partial v}{\partial \eta}. \tag{1.4.2.3}$$

Substituting into the transport equation yields

$$\frac{\partial v}{\partial \eta} + c\frac{\partial v}{\partial \xi} = 0, \tag{1.4.2.4}$$

which is invariant under the scaling transformations associated with the optimal subalgebra, demonstrating the equation's reduced form.

1.4.3 Infinitesimal Transformation

An infinitesimal transformation represents a minute change in variables or parameters under a Lie group, crucial for analyzing the behavior of differential equations through symmetry methods. For example, consider a Lie group with transformations

$$\bar{x} = e^{\varepsilon}x, \quad \bar{y} = e^{-\varepsilon}y. \tag{1.4.3.1}$$

The infinitesimal generators are derived as

$$\frac{\partial \bar{x}}{\partial \varepsilon} = e^{\varepsilon}x, \quad \left.\frac{\partial \bar{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = x, \tag{1.4.3.2}$$

treating x as constant. Similarly,

$$\frac{\partial \bar{y}}{\partial \varepsilon} = -e^{-\varepsilon} y, \quad \left.\frac{\partial \bar{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = -y, \tag{1.4.3.3}$$

treating y as constant. Thus, the infinitesimal generators are

$$X(x, y) = x, \quad Y(x, y) = -y, \tag{1.4.3.4}$$

which characterize the symmetry transformations of the system.

Chapter 2

LITERATURE REVIEW

This literature review examines critical advancements in Lie symmetry methods for solving non-linear partial differential equations (PDEs), highlighting their evolution from theoretical foundations to computational applications.

The selected studies, spanning from the 19th century to recent decades, demonstrate the progressive development of symmetry-based techniques for analyzing complex non-linear PDEs. Each contribution is presented with its key innovation and relevance to the field, emphasizing foundational and impactful works.

1. **S. Lie (1870):** In 1870, Lie introduced continuous transformation groups, fundamentally changing how differential equations are solved. His seminal work, as seen in "Vorlesungen über Differentialgleichungen," used infinitesimal transformations to reduce equations like $\frac{dy}{dx} = f(y)$ to quadratures. Collaborating with Felix Klein, Lie integrated geometric insights, establishing a systematic approach for deriving invariant solutions. This foundational contribution remains essential for applying Lie symmetry analysis to non-linear PDEs, providing the basis for modern symmetry methods [1].

2. **E. Noether (1918):** In 1918, Noether's work revolutionized mathematical physics by linking symmetries to conservation laws through Noether's theorem. For a Lagrangian $L(q, \dot{q}, t)$ invariant under time translation, she derived the conservation of

energy, $e = \dot{q}(\frac{\partial L}{\partial \dot{q}} - L)$. This theorem is pivotal for non-linear PDEs, connecting symmetries to physical properties like energy or momentum conservation, and it continues to guide the derivation of conservation laws in symmetry-based analyses [2].

3. **E. Cartan (1930):** In 1930, Cartan advanced Lie symmetry theory by introducing the method of moving frames, which facilitated the study of differential invariants on manifolds. His geometric approach simplified the classification of Lie algebras, making it easier to apply symmetry reductions to complex non-linear PDEs. Cartan's work provided a robust framework for analyzing PDE structures, enhancing the geometric understanding of symmetries [3].

4. **P. Olver (1980):** In 1980, Olver's work on modernized Lie symmetry analysis by developing algorithms for computing symmetry generators, as detailed in "Applications of Lie Groups to Differential Equations." His innovations made symmetry methods accessible and practical for non-linear PDEs, streamlining the derivation of invariant solutions. Olver's algorithms are a cornerstone of modern symmetry analysis, enabling efficient handling of complex equations [4].

5. **J.D. Logan and J.D.J. Perez (1980):** In 1980, Logan and Perez applied group theory to study wave interactions in reactive fluids, deriving invariant solutions for non-linear PDEs. Their work elucidated wave dynamics in complex systems, offering insights into non-linear wave propagation. This study is relevant for symmetry-based approaches to PDEs governing physical phenomena like wave interactions [5].

6. **J.P. Vishwakarma and G. Nath (2007):** In 2007, Vishwakarma and Nath investigated self-similar flows behind shock waves in non-ideal gases, using symmetry transformations to reduce PDEs to ODEs. Their methodology advanced the understanding of nonlinear wave behavior in gas dynamics, providing a model for symmetry reductions in nonlinear PDEs. This work is significant for its application to physical systems [6].

7. **J.P. Vishwakarma, A.K. Maurya and K.K. Singh (2007):** In 2007, Maurya explored magnetogasdynamic shock waves in non-ideal gases, using symmetry methods to simplify coupled non-linear PDEs. Their solutions for systems with magnetic fields offer insights into handling complex PDEs with external forces, relevant for symmetry-based reductions [10].

8. **G. Nath and J.P. Vishwakarma (2008):** In 2008, Nath and Vishwakarma derived solutions for shock waves in non-ideal gas mixtures using symmetry techniques. Their approach enhanced the analysis of nonlinear wave dynamics in multiphase systems, making it relevant for studying PDEs in diverse physical contexts [7].

9. **J.P. Vishwakarma and G. Nath (2012):** Vishwakarma and Nath's 2012 study combined symmetry methods with numerical simulations to analyze shock waves in dusty gases. This hybrid approach improved solution accuracy for complex nonlinear PDEs, marking a significant advancement by integrating analytical and computational techniques. Their work is particularly impactful for its practical applicability [8].

10. **M. Chadha and J. Jena (2014):** In 2014, Chadha and Jena studied gas-particle mixtures, deriving self-similar solutions via symmetry methods to reduce PDEs to ODEs for numerical solution. Their approach enhanced computational efficiency in fluid mechanics, supporting the analysis of non-linear PDEs in multiphase flows [11].

11. **G. Nath and J.P. Vishwakarma (2014):** Also in 2014, Nath and Vishwakarma investigated unsteady flows in non-ideal gases with magnetic fields, deriving similarity solutions via Lie symmetries. Their work advanced the study of nonlinear PDEs influenced by external forces, offering insights into magnetized systems [9].

12. **A. Devi and G. Nath (2021):** In 2021, Devi and Nath applied local Lie symmetry analysis to simplify PDEs for non-Newtonian fluid flows, deriving invariant solutions to reduce complexity. Their work advanced the analysis of non-linear PDEs in complex fluid dynamics, demonstrating the versatility of symmetry methods [12].

Chapter 3

LIE GROUP OF INVARIANCE METHOD

Lie symmetry analysis is a robust mathematical approach for studying partial differential equations (PDEs) by identifying transformations that preserve their form. These symmetries facilitate the simplification of PDEs, derivation of exact solutions, and identification of conserved quantities.

3.1 Methodology

Here is a stepwise explanation of the Lie group of invariance method to obtain exact solutions of the PDEs.

Step1. Identify the PDE and its independent and dependent variables. For example, consider the heat equation: $u_t = u_{xx}$, with independent variables t (time), x (space), and dependent variable u (temperature). The infinitesimal generator for a one-parameter Lie group of transformations:

$$\mathbf{v} = \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}, \tag{3.1.1}$$

where ξ, τ and ϕ are unknown functions that represent infinitesimal generator.

Step 2. Since PDEs involve derivatives, extend the generator to act on derivatives up to the PDE's order. For the second-order heat equation, prolong $\mathbf{v}$ to include transformations of u_x, u_t, and u_{xx} . The prolongation formula, based on the chain rule, ensures consistency with the PDE's differential structure.

Step 3. Apply the invariance condition to ensure the PDE remains unchanged under the prolonged generator:

$$\mathbf{v}^{(n)}(u_t - u_{xx}) = 0 \quad \text{whenever} \quad u_t - u_{xx} = 0, \tag{3.1.2}$$

where $\mathbf{v}^{(n)}$ is the n-th order prolonged generator. This generates a system of partial differential equations called determining equations for ξ, τ and ϕ .

Step 4. Solve the determining equations to find explicit expressions for ξ, τ and ϕ, which define the Lie algebra of symmetries.
For the heat equation, The Lie algebra of infinitesimal symmetries are spanned by the following vector fields:

$$\mathbf{v}_1 = \frac{\partial}{\partial x}, \quad \mathbf{v}_2 = \frac{\partial}{\partial t}, \quad \mathbf{v}_3 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}. \tag{3.1.3}$$

For complex PDEs, computational tools like Maple may be used to solve these equations efficiently.

Step 5. Form a commutative table to analyze the Lie algebra's structure. For vector fields $\mathbf{v}_i$ and $\mathbf{v}_j$, compute the Lie bracket:

$$[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_i\mathbf{v}_j - \mathbf{v}_j\mathbf{v}_i. \tag{3.1.4}$$

The table lists $\mathbf{v}_i, \mathbf{v}_j$, indicating whether the algebra is Abelian (brackets vanish) or non-Abelian. For the heat equation, the table may show non-zero brackets for scaling symmetries, revealing the algebra's complexity.

Step 6. Compute the adjoint representation to understand symmetry interactions. The adjoint action of $\mathbf{v}_i$ on $\mathbf{v}_j$ is:

$$\mathrm{Ad}(\mathbf{v}_i, \mathbf{v}_j) = [\mathbf{v}_i, \mathbf{v}_j]\varepsilon, \tag{3.1.5}$$

where ε is the group parameter. The adjoint table lists these actions, aiding in classifying the Lie algebra and identifying subalgebras for solution derivation.

Step 7. Use the generators to find invariant solutions by solving the characteristic equations:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}. \tag{3.1.6}$$

For a generator like $\mathbf{v}_1 = \frac{\partial}{\partial x}$, the invariant solution is $u = u(t)$. Substitute into the PDE to reduce it to an ODE, e.g., ($u_t = 0$) for the heat equation, yielding constant solution.

Step 8. Identify an optimal system of subalgebras using the commutative and adjoint tables. This involves classifying one- and two-dimensional subalgebras to avoid redundant solutions. For each subalgebra, compute invariant solutions, ensuring a minimal set of distinct solutions. For the heat equation, this may yield solutions like $u = e^{-t}\cos(x)$ from scaling subalgebras.

Step 9. Solve the reduced ODEs from invariant solutions, subject to initial or boundary conditions, to obtain exact solutions. For the heat equation, solutions like

$$u = c_1 + c_2 t \qquad \text{or} \qquad u = e^{-t}\cos(x) \tag{3.1.7}$$

may emerge, corresponding to specific generators. These solutions are exact and reflect the PDE's invariance properties.

Step 10. For PDEs with more than two independent variables, the method applies iteratively (e.g., $u_t = u_{xx} + u_{yy}$ in three variables (t, x, y)). Compute symmetries for subsets of variables, reduce the PDE step-by-step, and derive invariant solutions for each subalgebra. The number of iterations depends on the number of independent variables, potentially reducing the PDE to ODEs or simpler PDEs after each step.

3.2 Application of Lie Group of Invariance Method

In this subsection, we have solved a non-linear (2+1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff (nmCBS) system derived from the truncated Painlevé expansion [13]

$$u_y + u_{xxt} - 4u^2u_t - 4u_x \int uu_t dx = 0, \tag{3.2.1}$$

or equivalently

$$\begin{aligned} &u_y + u_{xxt} - 4u^2u_t - 4u_xv = 0, \\ &uu_t - v_x = 0. \end{aligned} \tag{3.2.2}$$

Related Work

Here, we have provided rich literature available for the $(2 + 1)$-dimensional nmCBS system. Wazwaz [14] employed direct algebraic methods to derive soliton and traveling wave solutions for the modified KdV-Calogero–Bogoyavlenskii–Schiff equation. Moatimid et al. [15] used Lie symmetry and singular manifold methods to obtain exact analytical solutions for the Calogero-Bogoyavlenskii-Schiff equation, applicable to the nmCBS due to its structural similarity. Al-Amin et al. [16] applied the enhanced auxiliary equation method to construct solitary wave solutions, including kink and rational solutions, for the nmCBS equation. Roshid [17] utilized the Hirota bilinear method to derive multiple soliton solutions for the nmCBS equation, analyzing their interaction dynamics. Cinar et al. [18] implemented the new Kudryashov method to obtain exact traveling wave solutions for the nmCBS equation, relevant to fluid mechanics and plasma physics. Liu et al. [19] investigated the negative-order modified Calogero–Bogoyavlenskii–Schiff (nmCBS) equation, using residual symmetry analysis and the consistent Riccati expansion method to derive multiple residual symmetries and soliton-cnoidal wave interaction solutions. In contrast, the present work applies Lie symmetry analysis to derive group-invariant solutions and an optimal system for the (2+1)-dimensional nmCBS system, a novel approach not fully explored in the above-mentioned studies.

Lie symmetry analysis of $(2+1)$-dimensional nmCBS system

Here, we aim to elucidate the fundamental concepts for deriving infinitesimal symmetries and the Lie algebra of the Eq. (3.2.2) through the Lie group theory. For (2+1)-dimensional nmCBS system, one-parameter (ϵ) Lie group of transformations is defined as follows:

$$\begin{aligned}
\bar{x} &= x + \epsilon\xi_x(x, y, t, u, v) + o(\epsilon^2), \\
\bar{y} &= y + \epsilon\xi_y(x, y, t, u, v) + o(\epsilon^2), \\
\bar{t} &= t + \epsilon\xi_t(x, y, t, u, v) + o(\epsilon^2), \\
\bar{u} &= u + \epsilon\eta_u(x, y, t, u, v) + o(\epsilon^2), \\
\bar{v} &= v + \epsilon\eta_v(x, y, t, u, v) + o(\epsilon^2).
\end{aligned} \tag{3.2.3}$$

Here, $\epsilon << 1$ represents a small parameter. The infinitesimals associated with the variables $x, y, t, u,$ and v are denoted by the functions $\xi_x, \xi_y, \xi_t, \eta_u, \eta_v$ respectively, and are to be determined. The Lie algebra for the system (3.2.2) is generated by a vector field expressed in the following form:

$$\begin{aligned}
\mathbf{V} =& \xi_x(x, y, t, u, w)\frac{\partial}{\partial x} + \xi_y(x, y, t, u, w)\frac{\partial}{\partial y} + \xi_t(x, y, t, u, w)\frac{\partial}{\partial t} \\
&+ \eta_u(x, y, t, u, w)\frac{\partial}{\partial u} + \eta_v(x, y, t, u, w)\frac{\partial}{\partial v}.
\end{aligned} \tag{3.2.4}$$

The vector field (3.2.4) defines a symmetry for Eq. (3.2.2), requiring the generator $\mathbf{V}$ to satisfy the invariance conditions:

$$\begin{aligned}
Pr^{(1)}\mathbf{V}(\Lambda_1)\big|_{\Delta_1=0} &= 0, \\
Pr^{(3)}\mathbf{V}(\Lambda_2)\big|_{\Delta_2=0} &= 0,
\end{aligned} \tag{3.2.5}$$

where $\Lambda_1 = uu_t - v_x$ and $\Lambda_2 = u_y + u_{xxt} - 4u^2u_t - 4u_xv = 0$. Here, $Pr^{(1)}$ and $Pr^{(3)}$ denote the first and third prolongations of $\mathbf{V}$, respectively, defined as follows:

$$Pr^{(1)}\mathbf{V} = \mathbf{V} + \eta_u^x \frac{\partial}{\partial u_x} + \eta_u^y \frac{\partial}{\partial u_y} + \eta_u^t \frac{\partial}{\partial u_t} + \eta_v^x \frac{\partial}{\partial v_x} + \eta_v^y \frac{\partial}{\partial v_y} + \eta_v^t \frac{\partial}{\partial v_t}, \tag{3.2.6}$$

$$\begin{aligned}
Pr^{(3)}\mathbf{V} = \mathbf{V} &+ \eta_u^x \frac{\partial}{\partial u_x} + \eta_u^y \frac{\partial}{\partial u_y} + \eta_u^t \frac{\partial}{\partial u_t} + \eta_v^x \frac{\partial}{\partial v_x} + \eta_v^y \frac{\partial}{\partial v_y} + \eta_v^t \frac{\partial}{\partial v_t} \\
&+ \eta_u^{xx} \frac{\partial}{\partial u_{xx}} + \eta_u^{xy} \frac{\partial}{\partial u_{xy}} + \eta_u^{xt} \frac{\partial}{\partial u_{xt}} + \eta_u^{yy} \frac{\partial}{\partial u_{yy}} + \eta_u^{yt} \frac{\partial}{\partial u_{yt}} + \eta_u^{tt} \frac{\partial}{\partial u_{tt}} \\
&+ \eta_v^{xx} \frac{\partial}{\partial v_{xx}} + \eta_v^{xy} \frac{\partial}{\partial v_{xy}} + \eta_v^{xt} \frac{\partial}{\partial v_{xt}} + \eta_v^{yy} \frac{\partial}{\partial v_{yy}} + \eta_v^{yt} \frac{\partial}{\partial v_{yt}} + \eta_v^{tt} \frac{\partial}{\partial v_{tt}} \\
&+ \eta_u^{xxx} \frac{\partial}{\partial u_{xxx}} + \eta_u^{xxy} \frac{\partial}{\partial u_{xxy}} + \eta_u^{xxt} \frac{\partial}{\partial u_{xxt}} + \eta_u^{xyy} \frac{\partial}{\partial u_{xyy}} + \eta_u^{xyt} \\
&+ \frac{\partial}{\partial u_{xyt}} + \eta_u^{xtt} \frac{\partial}{\partial u_{xtt}} + \eta_u^{yyy} \frac{\partial}{\partial u_{yyy}} + \eta_u^{yyt} \frac{\partial}{\partial u_{yyt}} + \eta_u^{ytt} \frac{\partial}{\partial u_{ytt}} + \eta_u^{ttt} \frac{\partial}{\partial u_{ttt}} \\
&+ \eta_v^{xxx} \frac{\partial}{\partial v_{xxx}} + \eta_v^{xxy} \frac{\partial}{\partial v_{xxy}} + \eta_v^{xxt} \frac{\partial}{\partial v_{xxt}} + \eta_v^{xyy} \frac{\partial}{\partial v_{xyy}} + \eta_v^{xyt} \frac{\partial}{\partial v_{xyt}} + \eta_v^{xtt} \frac{\partial}{\partial v_{xtt}} \\
&+ \eta_v^{yyy} \frac{\partial}{\partial v_{yyy}} + \eta_v^{yyt} \frac{\partial}{\partial v_{yyt}} + \eta_v^{ytt} \frac{\partial}{\partial v_{ytt}} + \eta_v^{ttt} \frac{\partial}{\partial v_{ttt}}.
\end{aligned}$$

The third prolongation is given by:

$$Pr^{(3)}\mathbf{V} = \mathbf{V} + \sum_{i\in\{u,v\}} \sum_{|J|\le 3} \eta_i^J \frac{\partial}{\partial u_i^J}, \tag{3.2.7}$$

where $u_u^J = u_J$, $u_v^J = v_J$, and $|J| = j_1 + j_2 + j_3 \le 3$. The coefficients η_i^J for $i \in \{u, v\}$ are defined as [4]:

$$\eta_i^J = D_J\left(\eta_i - \sum_{k=1}^{3} u_{i,k}\xi_k\right) + \sum_{k=1}^{3} u_{i,J+e_k}\xi_k,$$

with $u_{u,k} = u_k$, $u_{v,k} = v_k$, $u_1 = u_x$, $u_2 = u_y$, $u_3 = u_t$, $v_1 = v_x$, $v_2 = v_y$, $v_3 = v_t$, $\xi_1 = \xi_x$, $\xi_2 = \xi_y$, $\xi_3 = \xi_t$, and $D_J = D_x^{j_1} D_y^{j_2} D_t^{j_3}$.

Applying $Pr^{(1)}$ and $Pr^{(3)}$ to the (2+1)-dimensional nmCBS system (3.2.2) and enforcing the invariance conditions, we derive the invariant surface conditions (3.2.5) referred to as the determining equations.

By substituting the coefficients η_i^J into Eqs. (3.2.6) and (3.2.7), and utilizing equation

(3.2.2), we obtain an equation involving partial derivatives of $\xi_x, \xi_y, \xi_t, \eta_u$ and η_v. Equating the coefficients of the resulting monomials to zero yields a system of partial differential equations (PDEs). Solving this system determines the infinitesimal generators. These calculations can be performed manually or using software such as MAPLE or MATHEMATICA. In this study, infinitesimals were computed using MAPLE 15, as follows:

$$\begin{aligned}
\xi_x &= xc_3 + f_1(y),\\
\xi_y &= c_1 y + c_2,\\
\xi_t &= (c_1 - 2c_3)t + c_4,\\
\eta_u &= -uc_3,\\
\eta_v &= -\frac{1}{4} f_1^y(y) + (c_3 - c_1)v,
\end{aligned} \tag{3.2.8}$$

where $f_1(y)$ is the arbitrary functions of y, and $f_1^y(y)$ denotes the first derivative with respect to y. If we take $f_1(y) = c_5 y + c_6$, then we can write Eq. (3.2.8) as:

$$\begin{aligned}
\xi_x &= xc_3 + c_5 y + c_6,\\
\xi_y &= c_1 y + c_2,\\
\xi_t &= (c_1 - 2c_3)t + c_4,\\
\eta_u &= -uc_3,\\
\eta_v &= -\frac{1}{4} c_5 + (c_3 - c_1)v,
\end{aligned} \tag{3.2.9}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are the arbitrary constants. Corresponding to these infinitesimal generators we get the following vector fields:

$$\begin{aligned}
\mathbf{V} =&(c_3 x + c_5 y + c_6)\frac{\partial}{\partial x} + (c_1 y + c_2)\frac{\partial}{\partial y} + ((c_1 - 2c_3)t + c_4)\frac{\partial}{\partial t}\\
&- (c_3 u)\frac{\partial}{\partial u} + ((c_3 - c_1)v - \frac{1}{4}c_5)\frac{\partial}{\partial v}.
\end{aligned} \tag{3.2.10}$$

Now, corresponding to the different values of constants c_i (i =1,2,3,4,5,6), we consider the following different cases:

Case 1: $c_1 = 1, c_2 = c_3 = c_4 = c_5 = c_6 = 0$, we get vector field as:

$$\mathbf{V_1} = y\frac{\partial}{\partial y} + t\frac{\partial}{\partial t} - v\frac{\partial}{\partial v}. \tag{3.2.11}$$

Case 2: $c_2 = 1, c_1 = c_3 = c_4 = c_5 = c_6 = 0$, we get vector field as:

$$\mathbf{V_2} = \frac{\partial}{\partial y}. \tag{3.2.12}$$

Case 3: $c_3 = 1, c_1 = c_2 = c_4 = c_5 = c_6 = 0$, we get vector field as:

$$\mathbf{V_3} = x\frac{\partial}{\partial x} - 2t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}. \tag{3.2.13}$$

Case 4: $c_4 = 1, c_1 = c_2 = c_3 = c_5 = c_6 = 0$, we get vector field as:

$$\mathbf{V_4} = \frac{\partial}{\partial t}. \tag{3.2.14}$$

Case 5: $c_5 = 1, c_1 = c_2 = c_3 = c_4 = c_6 = 0$, we get vector field as:

$$\mathbf{V_5} = y\frac{\partial}{\partial x} - \frac{1}{4}v\frac{\partial}{\partial v}. \tag{3.2.15}$$

Case 6: $c_6 = 1, c_1 = c_2 = c_3 = c_4 = c_5 = 0$, we get vector field as:

$$\mathbf{V_6} = \frac{\partial}{\partial x}. \tag{3.2.16}$$

The set $[v_1, v_2, v_3, v_4, v_5$ and $v_6]$ is closed under Lie bracket. The commutation relations of Lie algebra between v_1, v_2, v_3, v_4, v_5 and v_6 are shown in Table 3.1.

Table 3.1: Commutation Table of Lie Algebra

$[\mathbf{V_i}, \mathbf{V_j}]$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$
$\mathbf{V}_1$	0	$-\mathbf{V}_2$	0	$-\mathbf{V}_4$	$\mathbf{V}_5$	0
$\mathbf{V}_2$	$\mathbf{V}_2$	0	0	0	$\mathbf{V}_6$	0
$\mathbf{V}_3$	0	0	0	$2\mathbf{V}_4$	$-\mathbf{V}_5$	$-\mathbf{V}_6$
$\mathbf{V}_4$	$\mathbf{V}_4$	0	$-2\mathbf{V}_4$	0	0	0
$\mathbf{V}_5$	$-\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_5$	0	0	0
$\mathbf{V}_6$	0	0	$\mathbf{V}_6$	0	0	0

In commutation Table 3.1, the $(i, j) - th$ entry represents the Lie bracket, defined as $[V_i, V_j] = V_iV_j - V_jV_i$. The table is skew-symmetric, featuring zero diagonal elements, and all V_i, for $i = 1, 2, 3, 4, 5, 6$, are linearly independent.

3.3 Optimal System of Subalgebras

Following the framework in [20 − 26], an optimal system of one-parameter subgroups consists of conjugacy-inequivalent subgroups such that any other subgroup is conjugate to exactly one in the list. Similarly, an optimal system of one-parameter subalgebras is formed if every one-parameter subalgebra is equivalent to a unique member of the list under the adjoint representation. Thus, classifying one-dimensional subalgebras is equivalent to classifying the orbits of the adjoint representation. In this study, we focus on the symmetry algebra of Eq. (3.2.2), spanned by the vector fields V_1, V_2, V_3, V_4, V_5 and V_6. Each one-dimensional subalgebra is defined by a nonzero vector of the form:

$$\mathbf{V} = c_1\mathbf{V}_1 + c_2\mathbf{V}_2 + c_3\mathbf{V_3} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}, \tag{3.3.1}$$

where c_i $(i = 1, 2, 3, 4, 5, 6)$ are arbitrary constants. We aim to simplify the coefficients c_i as much as possible by applying adjoint maps to $\mathbf{V}$. To achieve this, we compute the adjoint representation using the Lie series:

$$\mathrm{Ad}(\exp(\epsilon V_i))V_j = V_j - \epsilon[V_i, V_j] + \frac{\epsilon^2}{2}[V_i, [V_i, V_j]] - \cdots . \tag{3.3.2}$$

The adjoint representation of the vector fields for the Eq. (3.2.2) is presented in the table below.

Table 3.2: Adjoint Representation Table of Lie Algebra.

$Ad(exp(\epsilon_i\mathbf{V}_i)\mathbf{V}_j$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$
$\mathbf{V}_1$	$\mathbf{V_1}$	$e^{\epsilon_1}\mathbf{V_2}$	$\mathbf{V_3}$	$e^{\epsilon_1}\mathbf{V_4}$	$e^{-\epsilon_1}\mathbf{V_5}$	$\mathbf{V_6}$
$\mathbf{V}_2$	$\mathbf{V_1} - \epsilon_2\mathbf{V_2}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5} - \epsilon_2\mathbf{V_6}$	$\mathbf{V_6}$
$\mathbf{V}_3$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$e^{-2\epsilon_3}\mathbf{V_4}$	$e^{\epsilon_3}\mathbf{V_5}$	$e^{\epsilon_3}\mathbf{V_6}$
$\mathbf{V}_4$	$\mathbf{V_1} - \epsilon_4\mathbf{V_4}$	$\mathbf{V_2}$	$\mathbf{V_3} + 2\epsilon_4\mathbf{V_4}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$
$\mathbf{V}_5$	$\mathbf{V_1} + \epsilon_5\mathbf{V_5}$	$\mathbf{V_2} - \epsilon_5\mathbf{V_6}$	$\mathbf{V_3}$ - $\epsilon_5\mathbf{V_5}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$
$\mathbf{V}_6$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3} - \epsilon_6\mathbf{V_6}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$

3.3.1 Adjoint Transformation Matrix

Once the invariants of the Lie algebra are determined, the next step is to construct the general adjoint transformation matrix $\mathbf{A}$. This matrix is formed by multiplying the individual adjoint action matrices $\mathbf{A_1}$, $\mathbf{A_2}$, $\mathbf{A_3}$, $\mathbf{A_4}$, $\mathbf{A_5}$ and $\mathbf{A_6}$ corresponding to the adjoint actions of $\mathbf{V_1}$, $\mathbf{V_2}$, $\mathbf{V_3}$,$\mathbf{V_4}$, $\mathbf{V_5}$ and $\mathbf{V_6}$, respectively.

For convenience, an adjoint representation table is formulated, where the (i, j)-th entry represents $\mathrm{Ad}(\exp(\varepsilon \mathbf{V}_i))\mathbf{V}_j$, for $i, j = 1, 2, 3, 4, 5, 6$. The adjoint action of $\mathbf{V}_j$ on $\mathbf{V}_i$ can thus be obtained from Table 3.2.

Let ε_i, for $i = 1, 2, 3, 4, 5, 6$, be real constants, and define $g = e^{\varepsilon_i \mathbf{V}_i}$. Using the adjoint representation table (Table 3.2), we derive six adjoint matrices as follows:

$$\mathbf{A_1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & e^{\epsilon_1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{\epsilon_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & e^{-\epsilon_1} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A_2} = \begin{pmatrix} 1 & -\epsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\epsilon_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A_3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{-2\epsilon_3} & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{\epsilon_3} & 0 \\ 0 & 0 & 0 & 0 & 0 & e^{\epsilon_3} \end{pmatrix} \qquad \mathbf{A_4} = \begin{pmatrix} 1 & 0 & 0 & -\epsilon_4 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2\epsilon_4 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{A_5} = \begin{pmatrix} 1 & 0 & 0 & 0 & \epsilon_5 & 0 \\ 0 & 1 & 0 & 0 & 0 & -\epsilon_5 \\ 0 & 0 & 1 & 0 & -\epsilon_5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{A_6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -\epsilon_6 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Constructing an optimal system of subalgebras for a Lie algebra is a complex task. The optimal system is obtained by solving a system of algebraic equations, with equivalent Lie subalgebras identified through the adjoint action on the set of subalgebras. Our goal is to simplify the general expression of $\mathbf{V}$, given in Eq. (3.3.1), where $c_1, c_2, c_3, c_4, c_5, c_6$ are arbitrary real constants, as previously noted. We represent $\mathbf{V}$ as a column vector with entries $c_1, c_2, c_3, c_4, c_5, c_6$. The general adjoint transformation matrix $\mathbf{A}$ is then obtained as the product:

$$\mathbf{A}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) = \mathbf{A_6 A_5 A_4 A_3 A_2 A_1}, \tag{3.3.1.1}$$

which is

$$\mathbf{A} = \begin{pmatrix} 1 & -e^{\epsilon_1}\epsilon_2 & 0 & -e^{\epsilon_1 - 2\epsilon_3}\epsilon_4 & e^{-\epsilon_1+\epsilon_3}\epsilon_5 & -e^{\epsilon_3}\epsilon_2\epsilon_5 \\ 0 & e^{\epsilon_1} & 0 & 0 & 0 & -e^{\epsilon_3}\epsilon_5 \\ 0 & 0 & 1 & 2e^{\epsilon_1-2\epsilon_3}\epsilon_4 & -e^{-\epsilon_1+\epsilon_3}\epsilon_5 & e^{\epsilon_3}(\epsilon_2\epsilon_5 - \epsilon_6) \\ 0 & 0 & 0 & e^{\epsilon_1-2\epsilon_3}\epsilon_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & e^{-\epsilon_1+\epsilon_3}\epsilon_2 & -e^{\epsilon_3}\epsilon_2 \\ 0 & 0 & 0 & 0 & 0 & -e^{\epsilon_3} \end{pmatrix} \tag{3.3.1.2}$$

3.3.2 Optimal Subalgebra

To construct a one-dimensional optimal system, consider $\mathbf{V} = \sum_{i=1}^{6} c_i \mathbf{V_i}$ and $M = \sum_{i=1}^{6} q_i \mathbf{V_i}$ as elements of the Lie algebra L_6. Substituting $\mathbf{V}$ and M into the adjoint transformation equation for system (3.2.2), given by:

$$(q_1, q_2, q_3, q_4, q_5, q_6) = (c_1, c_2, c_3, c_4, c_5, c_6) \cdot A, \tag{3.3.2.1}$$

yields the values of ε_i, for $i = 1, 2, 3, 4, 5, 6$. The matrix $\mathbf{A}(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6)$ transforms $\mathbf{V}$ as:

$$\begin{aligned}
B(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) \cdot \mathbf{V} =& c_1 \mathbf{V_1} + e^{\epsilon_1}(-\epsilon_2 c_1 + c_2)\mathbf{V_2} + c_3 \mathbf{V_3} \\
&+ e^{\epsilon_1 - 2\epsilon_3}(-c_1\epsilon_4 + 2c_3\epsilon_4 + c_4)\mathbf{v_4} + e^{-\epsilon_1+\epsilon_3}(c_1\epsilon_5 - c_3\epsilon_5 + c_5)\mathbf{V_5} \\
&+ e^{\epsilon_3}(-c_1\epsilon_2\epsilon_5 - c_2\epsilon_5 + c_3\epsilon_2\epsilon_5 - c_3\epsilon_6 - c_5\epsilon_2 + c_6)\mathbf{V_6}.
\end{aligned} \tag{3.3.2.2}$$

By definition, $\mathbf{V}$ and $A(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6) \cdot \mathbf{V}$ generate equivalent one-dimensional Lie subalgebras for any $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6$. This allows flexibility in choosing ε_i values to obtain a simpler representative of the equivalence class of $\mathbf{V}$. To classify the one-dimensional Lie subalgebras of Eq. (3.2.2), four cases are considered based on the degree of the invariants.

Case 1: $c_1, c_3 = l$, where $l \neq 0$ is an arbitrary real constant. Choosing a representative element $\tilde{\mathbf{V_1}} = \mathbf{V}_1 + c_2\mathbf{V}_2 + l\mathbf{V_3} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}$ and substituting $q_1 = 1$, $q_2 = q_4 = q_5 = q_6 = 0$, and $q_3 = l$ into Eq. (3.3.2.2), we derive the solution:

$$\begin{aligned}
\varepsilon_2 &= c_2, \\
\varepsilon_4 &= -\frac{c_4}{1+2l}, \\
\varepsilon_5 &= \frac{c_5}{l-1}, \\
\varepsilon_6 &= \frac{1}{l}\left(\frac{(c_2c_5)(l-2)}{l-1} - c_2c_5 + c_6\right).
\end{aligned} \tag{3.3.2.3}$$

Thus, the adjoint map $\mathrm{Ad}(\exp(\varepsilon_2\mathbf{V}_2))$ eliminates the $\mathbf{V_2}, \mathbf{V_4}, \mathbf{V_5}, \mathbf{V_6}$, coefficient from $\tilde{V}$, which is equivalent to

$$\tilde{\mathbf{V_1}} = \mathbf{V_1} + c_3\mathbf{V_3}. \tag{3.3.2.4}$$

Case 2: $c_1, c_3 = 0$, is an arbitrary real constant. Choosing a representative element $\tilde{\mathbf{V_1}} = \mathbf{V}_1 + c_2\mathbf{V}_2 + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}$ and substituting $q_1 = 1$, $q_2 = q_3 = q_4 = q_5 = q_6 = 0$, into Eq. (3.3.2.2), we derive the solution:

$$
\begin{aligned}
\varepsilon_2 &= c_2,\\
\varepsilon_4 &= c_4,\\
\varepsilon_5 &= -c_5.
\end{aligned}
\tag{3.3.2.5}
$$

Thus, the adjoint map $\mathrm{Ad}(\exp(\varepsilon_2 X_2))$ eliminates the $\mathbf{V_2}, \mathbf{V_4}, \mathbf{V_5}$ coefficient from $\tilde{V}$, which is equivalent to

$$\tilde{\mathbf{V_2}} = \mathbf{V_1} + c_6\mathbf{V_6}. \tag{3.3.2.6}$$

Case 3: $c_1 = 0, c_3 = 1$, is an arbitrary real constant. Choosing a representative element $\tilde{\mathbf{V_1}} = c_2\mathbf{V}_2 + \mathbf{V_3} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}$ and substituting $q_3 = 1$, $q_1 = q_2 = q_4 = q_5 = q_6 = 0$, into Eq. (3.3.2.2), we derive the solution:

$$
\begin{aligned}
\varepsilon_4 &= -\frac{1}{2}c_4,\\
\varepsilon_5 &= c_5,\\
\varepsilon_6 &= -c_6.
\end{aligned}
\tag{3.3.2.7}
$$

Thus, the adjoint map $\mathrm{Ad}(\exp(\varepsilon_2 \mathbf{V}_2))$ eliminates the $\mathbf{V_4}, \mathbf{V_5}, \mathbf{V_6}$ coefficient from $\tilde{V}$, which is equivalent to

$$\tilde{\mathbf{V_3}} = c_2\mathbf{V_2} + \mathbf{V_6}. \tag{3.3.2.8}$$

Case 4: $c_1 = 0, c_3 = 0$, is an arbitrary real constant. Choosing a representative element $\tilde{\mathbf{V_1}} = c_2\mathbf{V}_2 + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}$ and substituting $q_1 = q_2 = q_3 = q_4 = q_5 = q_6 = 0$, into Eq. (3.3.2.2). Thus, $\tilde{V}$ is equivalent to

$$\tilde{\mathbf{V_4}} = c_2\mathbf{V_2} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}. \tag{3.3.2.9}$$

To summarize, a one-dimensional optimal system of subalgebras of (2+1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff (nmCBS) system is given by $\left(\tilde{\mathbf{V_1}}, \tilde{\mathbf{V_2}}, \tilde{\mathbf{V_3}}, \tilde{\mathbf{V_4}}\right)$, where

$$
\begin{aligned}
&\tilde{\mathbf{V_1}} = \mathbf{V_1} + c_3\mathbf{V_3}, && \tilde{\mathbf{V_2}} = \mathbf{V_1} + c_6\mathbf{V_6},\\
&\tilde{\mathbf{V_3}} = c_2\mathbf{V_2} + \mathbf{V_6}, && \tilde{\mathbf{V_4}} = c_2\mathbf{V_2} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}.
\end{aligned}
\tag{3.3.2.10}
$$

3.4 Symmetry Reductions and Invariant Solutions of (2+1)-dimensional nmCBS System

In above section (3.3) we have constructed a one-dimensional optimal system of Lie subalgebras for Eq. (3.2.2). In this section, we use the optimal algebra (3.3.2.10) to derive several Lie symmetry reductions and group invariant solutions for equation (3.2.2) in the following manner:

Case 1: $\tilde{\mathbf{V_1}} = \mathbf{V_1} + c_3\mathbf{V_3}$.

Take $c_3 = 1$. The characteristic equations corresponding to this subalgebra are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{-t} = \frac{du}{-u} = \frac{dv}{0}. \tag{3.4.1}$$

Integrating Eq. (3.4.1) we obtain the similarity variables and the corresponding similarity forms as follows:

$$\mathbf{X} = xt, \mathbf{Y} = yt, u = t\mathbf{U}(\mathbf{X}, \mathbf{Y}), v = \mathbf{V}(\mathbf{X}, \mathbf{Y}). \tag{3.4.2}$$

Here, the new independent variables $\mathbf{X}$, $\mathbf{Y}$ are named as similarity variables, and the new dependent variables are $\mathbf{U}$ and $\mathbf{V}$ are named as similarity function. So Eq. (3.2.2) in terms of new variables is:

$$\begin{aligned}
&\mathbf{U_Y} + 3\mathbf{U_X X} + \mathbf{XU_X XY} + \mathbf{YU_X XY} - 4\mathbf{U^3} - 4\mathbf{XU^2U_X} - 4\mathbf{YU^2U_Y} - 4\mathbf{U_X V} = 0,\\
&\mathbf{U^2} + \mathbf{XU_X} + \mathbf{YU_Y} - \mathbf{V_X} = 0.
\end{aligned} \tag{3.4.3}$$

Again, utilizing the Lie symmetry method on Eq. (3.4.3), we get the following infinitesimals:

$$\begin{aligned}
&\xi_{\mathbf{X}} = c_1 Y, && \xi_{\mathbf{Y}} = 0,\\
&\eta_{\mathbf{U}} = 0, && \eta_{\mathbf{V}} = -\tfrac{1}{4}c_1,
\end{aligned} \tag{3.4.4}$$

where c_1 is arbitrary constant. Take $c_1 = 1$. Using equation (3.4.3) and (3.4.4), the characteristic equations are

$$\frac{d\mathbf{X}}{\mathbf{Y}} = \frac{d\mathbf{Y}}{0} = \frac{d\mathbf{U}}{0} = \frac{d\mathbf{V}}{-\frac{1}{4}}. \tag{3.4.5}$$

Integrating Eq. (3.4.5), we get:

$$\mathbf{P} = \mathbf{Y}, \mathbf{U} = \mathbf{U}(\tilde{\mathbf{P}}), \mathbf{V} = \mathbf{V}(\tilde{\mathbf{P}}). \tag{3.4.6}$$

On substituting equation (3.4.6) into (3.4.3), we obtained the reduced system of ODEs as follow:

$$\begin{aligned} &\tilde{\mathbf{U}_\mathbf{P}} - 4\tilde{\mathbf{U}^3} - 4\mathbf{P}\tilde{\mathbf{U}^2}\tilde{\mathbf{U}_\mathbf{P}} = 0, \\ &4\mathbf{P}\tilde{\mathbf{U}^2} + 4\mathbf{P^2}\tilde{\mathbf{U}_\mathbf{P}} - \tilde{\mathbf{v}} = 0. \end{aligned} \tag{3.4.7}$$

The general solution of the system (3.4.7) is given as:

$$\tilde{\mathbf{U}} = \frac{c_1+\sqrt{c_1^2-\mathbf{P}}}{\mathbf{2P}}, \qquad \tilde{\mathbf{V}} = \frac{\mathbf{16P^2}\tilde{\mathbf{U}^3}-\mathbf{4P}\tilde{\mathbf{U}^2}}{\mathbf{4P}\tilde{\mathbf{U}^2}}. \tag{3.4.8}$$

It is the computed general solution of the system (3.4.7), however we obtained one particular solution by choose $c_1 = 1$. Now the invariant solution of the system (3.2.2) is given by:

$$\begin{aligned} u &= \frac{1+\sqrt{1+yt}}{2y}, \\ v &= \frac{-t^2x}{y} - \frac{(1+\sqrt{1+yt})^2(ty(1+y+2\sqrt{1+ty}) - 2(1+\sqrt{1+ty})}{y^3(8+3ty+8\sqrt{1+ty})}. \end{aligned} \tag{3.4.9}$$

Case 2: $\tilde{\mathbf{V_3}} = c_2\mathbf{V_2} + \mathbf{V_3}$.

Take $c_2 = 1$. The characteristic equations corresponding to this subalgebra are

$$\frac{dx}{x} = \frac{dy}{1} = \frac{dt}{-2t} = \frac{du}{-u} = \frac{dv}{v}. \tag{3.4.10}$$

Integrating Eq. (3.4.10) we obtain the similarity variables as:

$$\mathbf{X} = x^2t, \mathbf{Y} = 2y + ln(t), u = \sqrt{t}\mathbf{U}(\mathbf{X},\mathbf{Y}), v = \frac{\mathbf{V}(\mathbf{X},\mathbf{Y})}{\sqrt{t}}. \tag{3.4.11}$$

Here, the new independent variables **X**, **Y** are named as similarity variables, and the new dependent variables are **U** and **V** are named as similarity functions. So Eq. (3.2.2) in terms of new variables are:

$$
\begin{aligned}
&2\mathbf{U_Y} + 3\mathbf{U_X} + 12\mathbf{X}\mathbf{U_X}\mathbf{X} + 2\mathbf{U_X}\mathbf{Y} + 4\mathbf{X^2}\mathbf{U_X}\mathbf{XX} + 4\mathbf{X}\mathbf{U_X}\mathbf{XY} - 2\mathbf{U^3} \\
&- 4\mathbf{X}\mathbf{U^2}\mathbf{U_X} - 4\mathbf{U^2}\mathbf{U_Y} - 8\sqrt{\mathbf{X}}\mathbf{U_X}\mathbf{V} = 0, \\
&\mathbf{U^2} + 2\mathbf{X}\mathbf{U}\mathbf{U_X} + 2\mathbf{U}\mathbf{U_Y} - 4\sqrt{\mathbf{X}}\mathbf{V_X} = 0.
\end{aligned}
\tag{3.4.12}
$$

Again, utilizing the Lie symmetry method on Eq. (3.4.12), then we get the desired infinitesimals

$$
\begin{aligned}
&\xi_{\mathbf{X}} = c_2\sqrt{\mathbf{X}}\mathbf{e}^{\frac{\mathbf{Y}}{\mathbf{2}}}Y, && \xi_{\mathbf{Y}} = c_1, \\
&\eta_{\mathbf{U}} = 0, && \eta_{\mathbf{V}} = -\tfrac{1}{8}c_2\mathbf{e}^{\frac{\mathbf{Y}}{\mathbf{2}}}.
\end{aligned}
\tag{3.4.13}
$$

where c_1 and c_2 are arbitrary constants. Take $c_1 = 1, c_2 = 0$. Using Eqs. (3.4.12) and (3.4.13), the characteristic equations are

$$
\frac{d\mathbf{X}}{\mathbf{0}} = \frac{d\mathbf{Y}}{1} = \frac{d\mathbf{U}}{0} = \frac{d\mathbf{V}}{0}. \tag{3.4.14}
$$

Integrating Eq. (3.4.14) we get:

$$
\mathbf{P} = \mathbf{X}, \mathbf{U} = \mathbf{U}(\tilde{\mathbf{P}}), \mathbf{V} = \mathbf{V}(\tilde{\mathbf{P}}). \tag{3.4.15}
$$

On substituting Eq. (3.4.15) into (3.4.12), we obtain the reduced system of ODEs as follow:

$$
\begin{aligned}
&3\tilde{\mathbf{U}}_{\mathbf{P}} + 12\mathbf{P}\mathbf{U}_{\tilde{\mathbf{P}}}\mathbf{P} + 4\mathbf{P^2}\mathbf{U_P}\tilde{\mathbf{P}}\mathbf{P} - 2\tilde{\mathbf{U}}^{\mathbf{3}} - 4\mathbf{P}\tilde{\mathbf{U}}^{\mathbf{2}}\tilde{\mathbf{U}}_{\mathbf{P}} - 8\sqrt{\mathbf{P}}\tilde{\mathbf{U}}_{\mathbf{P}}\tilde{\mathbf{V}} = 0, \\
&\tilde{\mathbf{U}}^{\mathbf{2}} + 2\mathbf{P}\tilde{\mathbf{U}}\tilde{\mathbf{U}}_{\mathbf{P}} - 4\sqrt{\mathbf{P}}\tilde{\mathbf{V}}_{\mathbf{P}} = 0.
\end{aligned}
\tag{3.4.16}
$$

The general solution of the system (3.4.16) is given as:

$$
\tilde{\mathbf{U}} = 0, \qquad \tilde{\mathbf{V}} = c_1. \tag{3.4.17}
$$

Now, the invariant solution of the system (3.2.2) is given by :

$$
u = 0, \qquad v = \frac{c_1}{\sqrt{t}}. \tag{3.4.18}
$$

Case 3: $\tilde{\mathbf{V}_3} = c_2\mathbf{V_2} + c_4\mathbf{V_4} + c_5\mathbf{V_5} + c_6\mathbf{V_6}$

.

Take $c_2 = 1$. The characteristic equations corresponding to this subalgebra are

$$\frac{dx}{y+1} = \frac{dy}{1} = \frac{dt}{1} = \frac{du}{0} = \frac{dv}{-\frac{1}{4}}. \tag{3.4.19}$$

Integrating Eq. (3.4.19) we obtain the similarity variables as:

$$\mathbf{X} = x - \frac{y^2}{2} - y, \mathbf{T} = t - y, u = \mathbf{U}(\mathbf{X}, \mathbf{Y}), v = \frac{\mathbf{V}(\mathbf{X}, \mathbf{Y}) - \mathbf{y}}{\sqrt{4}}. \tag{3.4.20}$$

Here, the new independent variables $\mathbf{X}$, $\mathbf{Y}$ are named as similarity variables, and the new dependent variables $\mathbf{U}$ and $\mathbf{V}$ are named as similarity functions. So Eq. (3.2.2) in the terms of new variables is:

$$\begin{aligned} &\mathbf{U_X} + \mathbf{U_T} + -\mathbf{U_X}\mathbf{X}\mathbf{T} + 4\mathbf{U^2}\mathbf{U_T} + \mathbf{U_X}\mathbf{V} = 0, \\ &4\mathbf{U}\mathbf{U_T} - \mathbf{V_X} = 0. \end{aligned} \tag{3.4.21}$$

Again, utilizing the Lie symmetry method on Eq. (3.4.21), we get the desired infinitesimals as

$$\begin{aligned} &\xi_{\mathbf{X}} = c_1, && \xi_{\mathbf{T}} = f_1(\mathbf{T}), \\ &\eta_{\mathbf{U}} = 0, && \eta_{\mathbf{V}} = -(\mathbf{V} + \mathbf{1}) f_1^{\mathbf{T}}(\mathbf{T}). \end{aligned} \tag{3.4.22}$$

For simplicity take $f_1(\mathbf{T}) = c_2\mathbf{T}$, so the derivative of f_1 with respect to $\mathbf{T}$ is $f_1^{\mathbf{T}}(\mathbf{T}) = c_2$, where c_1 and c_2 are arbitrary constants. Take $c_1 = 0, c_2 = 1$. Using Eqs. (3.4.21) and (3.4.22), the characteristic equations are

$$\frac{d\mathbf{X}}{\mathbf{0}} = \frac{d\mathbf{T}}{t} = \frac{d\mathbf{U}}{0} = \frac{d\mathbf{V}}{-(v+1)}. \tag{3.4.23}$$

Integrating Eq. (3.4.23) we get:

$$\mathbf{P} = \mathbf{X}, \mathbf{U} = \tilde{\mathbf{U}(\mathbf{P})}, \mathbf{V} = \frac{\tilde{\mathbf{V}(\mathbf{P})}}{\mathbf{T}} - \mathbf{1}. \tag{3.4.24}$$

On substituting Eq. (3.4.24) into (3.4.21), we obtain the reduced system of ODEs as follow:

$$\frac{\tilde{\mathbf{U}_\mathbf{P}}\tilde{V}}{\mathbf{P}} = 0, \qquad \frac{\tilde{\mathbf{V}_\mathbf{P}}}{\mathbf{P}} = 0. \tag{3.4.25}$$

The general solution of the system (3.4.25) is given as:

$$\tilde{\mathbf{U}} = c_1, \qquad \tilde{\mathbf{V}} = c_2. \tag{3.4.26}$$

Now, the invariant solution of the system (3.2.2) is given by :

$$u = c_1, \qquad v = \frac{c_2 - 1}{4(t-y)} - \frac{y}{4}. \tag{3.4.27}$$

Chapter 4

RESULTS AND DISCUSSION

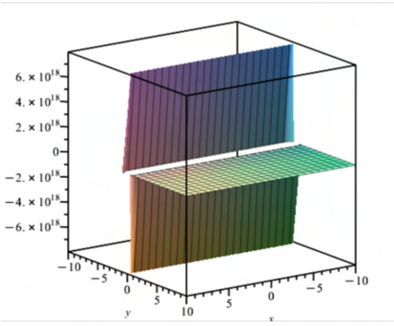
Graphical representation is an effective method of physically analyzing the nature of mathematical expressions. It offers visual evidence that aids in understanding the dynamics of the phenomena under consideration. This chapter focuses on the physical interpretation of the solutions obtained for the (2+1)-dimensional nmCBS system (3.2.2). These solutions are illustrated graphically using 3D, 2D and corresponding contour plots, presented in Figs. 1–10 through numerical simulations conducted with the software package "MAPLE." The obtained solutions involve a variety of arbitrary constants and functions, allowing for the depiction of diverse physical structures by choosing appropriate values. These solutions encompass multi-soliton, doubly soliton, single soliton, solitary waves, stationary waves, and the annihilation of various dynamical structures.

Solitons, also known as solitary waves, represent distinct wave structures consisting of solitary wave packets characterized by elastic scattering properties, enabling them to retain their shape even after interacting with each other. Solitons can be seen as the result of balancing non-linear and dispersion effects. They have numerous practical applications in fields such as plasma physics, optical lattices, stratified fluid flows, and biology.

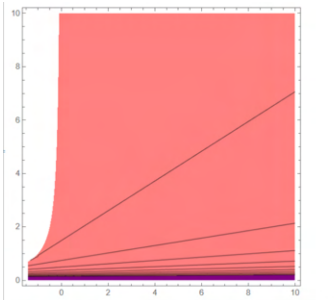
Here, we have plotted the solutions obtained in Case-1 and Case-3 only. Figure 4.1 depicts the graphical representation of solution u and v, given by Eq. (3.5.9). Figure 4.1(a) shows kink soliton wave profile for u in opposite directions. Figures 4.1(b), 4.1(c) and 4.1(d)

present the corresponding contour plot, 2D plots at $t = 2$ and at $y = 2$, respectively. Figure 4.1(e) shows singular soliton wave profile for v given in Eq. (3.5.9). Figures 4.1(f) and 4.1(g) present the corresponding contour plot and 2D plot at $x = 1$. Figure 4.1(h) shows singular soliton wave profile for v in Eq. (3.5.9). Figure 4.1(i) presents the corresponding contour plot in different ranges.

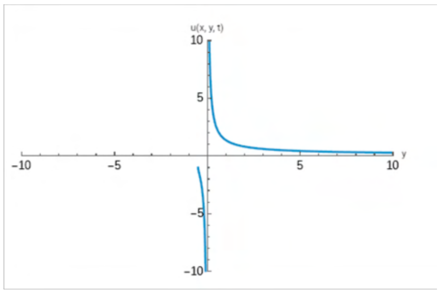
Figure 4.2 shows the graphical representation of the solution u and v, given by Eq. (3.4.27). Figure 4.2(a) shows constant soliton wave profile for u. Figures 4.2(b) and 4.2(c) present the corresponding contour plot and 2D plot at $x = 1$, respectively. Figure 4.2(d) shows multi-soliton wave profile for v. Figures 4.2(e) and 4.2(f) present the corresponding contour plot and 2D plot at $t = 0$, respectively. Figure 4.2(g) shows soliton wave profile for v in different ranges. Figures 4.2(h) and 4.2(i) present the corresponding contour plot and 2D plot at $t = 0$ in different ranges. Here, the values of arbitrary constants are considered as $c_1 = 1$ and $c_2 = 1$.



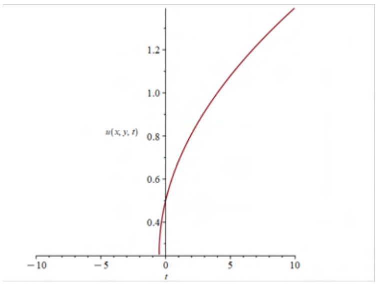
(a) 3D plot of sol. u



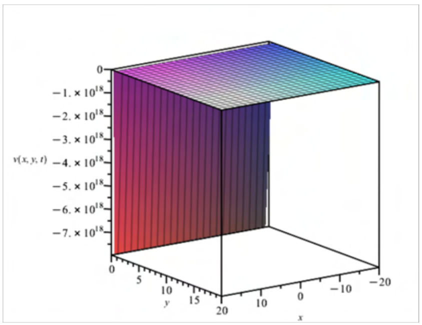
(b) Contour plot of sol. u



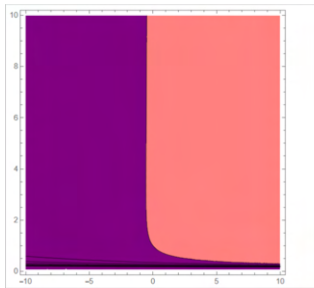
(c) 2D plot of sol. u at t =2



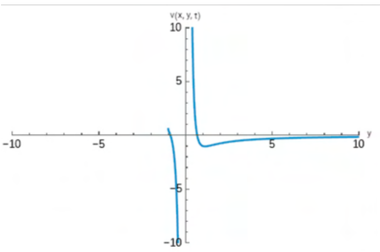
(d) 2D plot of sol. u at y = 2



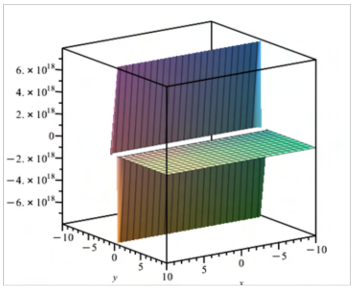
(e) 3D plot of sol. v



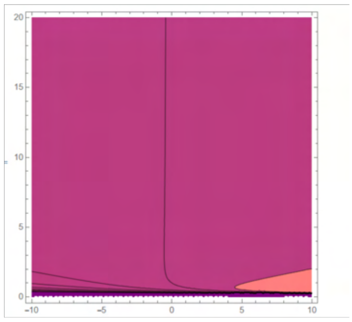
(f) Contour plot of sol. v



(g) 2D plot of sol. v at x =1

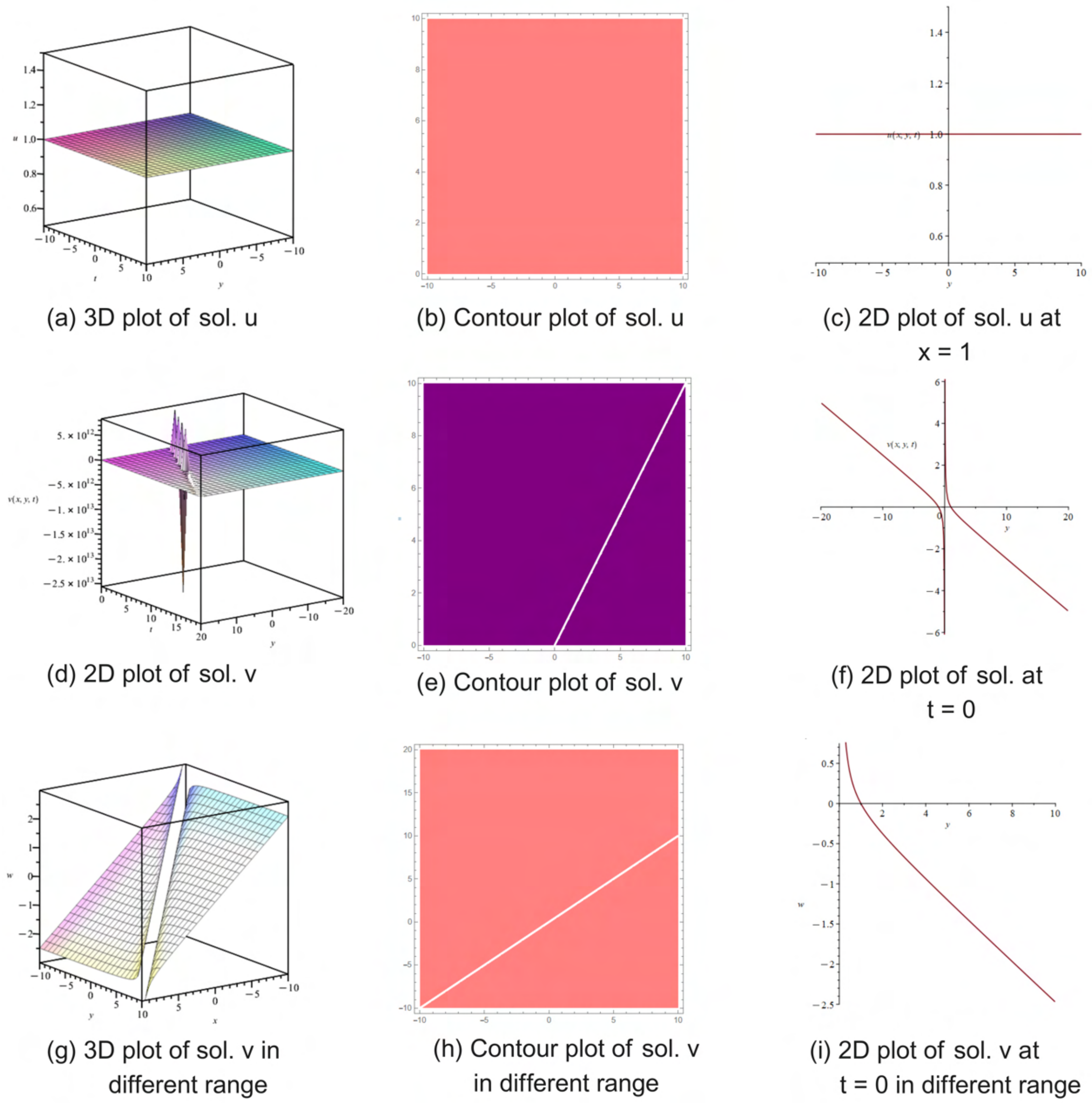


(h) 3D plot of sol. v in different range.



(i) Contour plot of sol. v in different range.

Figure 4.1: Solution profiles for Eq. (3.4.9).



(a) 3D plot of sol. u

(b) Contour plot of sol. u

(c) 2D plot of sol. u at x = 1

(d) 2D plot of sol. v

(e) Contour plot of sol. v

(f) 2D plot of sol. at t = 0

(g) 3D plot of sol. v in different range

(h) Contour plot of sol. v in different range

(i) 2D plot of sol. v at t = 0 in different range

Figure 4.2: Solution profiles for Eq. (3.4.27).

Chapter 5

SUMMARY AND FUTURE WORK

Summary

This study applies the Lie symmetry method to investigate the non-linear (2+1)-dimensional nmCBS system, a hyperbolic partial differential equation central to understanding soliton-like wave propagation in non-linear systems. The Lie symmetry approach involves identifying invariance properties of the equation under continuous group transformations, enabling the reduction of the PDE to simpler ordinary differential equations through symmetry-based transformations.

This methodology facilitates the derivation of exact solutions that capture the dynamical behavior of non-linear waves, such as solitons, under various physical conditions. By leveraging these symmetries, the study elucidates the underlying structure of the nmCBS equation, contributing to theoretical insights into wave phenomena with applications in fluid dynamics, optical solitons, and plasma physics, where hyperbolic PDEs model complex wave interactions.

Future work

Future investigations into hyperbolic partial differential equations and symmetry based methods can address several theoretical and practical gaps to enhance their applicability to non-linear wave phenomena. Incorporating real-world physical effects, such as viscosity, thermal variations, or external perturbations, could improve the modeling of wave dynamics in complex systems like fluids or optical media, necessitating adaptations of symmetry techniques to account for non-ideal conditions.

Extending analyses to multi-dimensional frameworks would enable a deeper understanding of intricate wave interactions, such as scattering or collision processes, by leveraging advanced symmetry reductions. Experimental validation of theoretical solutions in physical systems, such as wave propagation experiments, would strengthen the connection between models and real-world observations.

Additionally, adopting robust computational tools for visualization could enhance the interpretation of wave behaviors, addressing limitations in current graphical methods. Exploring fractional derivatives in non-linear wave equations could capture memory-dependent effects, requiring novel symmetry-based approaches to handle non-local dynamics, thus broadening the scope of wave propagation studies.

Bibliography

1. Lie, S. (1870). Lectures on Differential Equations with Known Infinitesimal Transformations. Leipzig: Teubner, 568, 1–200.

2. Noether, E. (1918). Invariante Variationsprobleme. Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, 1918, 235–257.

3. Cartan, É. (1930). La Méthode du Repère Mobile, la Théorie des Groupes Continus, et les Espaces Généralisés. Paris: Hermann, 224, 1–150.

4. Olver, P. J. (1986). Applications of Lie Groups to Differential Equations. New York: Springer-Verlag, 513, 1–401.

5. Logan, J. D., Perez, J. D. J. (1980). Similarity solutions for reactive shock hydrodynamics. SIAM Journal on Applied Mathematics, 39(3), 512–527.

6. Vishwakarma, J. P., Nath, G. (2007). Similarity solutions for unsteady flow behind an exponential shock in a dusty gas. Physica Scripta, 75(6), 829–838.

7. Nath, G., Vishwakarma, J. P. (2008). Propagation of a strong shock wave in a mixture of a non-ideal gas and small solid particles. Acta Astronautica, 62(2-3), 131–143.

8. Vishwakarma, J. P., Nath, G. (2012). A self-similar solution for unsteady adiabatic and isothermal flows behind the shock wave in a non-ideal gas using Lie group analysis. Meccanica, 47(5), 1207–1222.

9. Nath, G., Vishwakarma, J. P. (2014). Similarity solutions for the flow behind an exponential shock in an axisymmetric rotating non-ideal gas. Journal of Engineering Mathematics, 86(1), 147–162.

10. Vishwakarma, J. P., Maurya, A. K., and Singh, K. K. (2007). Self-similar adiabatic flow headed by a magnetogasdynamic cylindrical shock wave in a rotating non-ideal gas. Geophysical and Astrophysical Fluid Dynamics, 101(2), 155-168.

11. Chadha, M., Jena, J. (2014). Self-similar solutions for unsteady flow behind a shock wave in a gas-particle mixture. International Journal of Non-Linear Mechanics, 67, 76–85.

12. Devi, A., Nath, G. (2021). Lie symmetry analysis for the equations governing the flow of a non-Newtonian fluid with particles. Journal of Applied Mathematics and Computing, 65(1-2), 683–704.

13. Cheng, W., Qiu, D., Xu, T. (2020). Multiple residual symmetries and soliton-cnoidal wave interaction solution of the (2+1)-dimensional negative-order modified Calogero–Bogoyavlenskii–Schiff equation. The European Physical Journal Plus, 135(1), 15.

14. Wazwaz, A. M. (2018). Solutions for the modified KdV-Calogero–Bogoyavlenskii–Schiff equation. Applied Mathematics and Computation, 331, 434–441.

15. Moatimid, G. M., El-Shiekh, R. M., Al-Nowehy, A. G. (2013). Exact solutions for Calogero-Bogoyavlenskii-Schiff equation using symmetry method. Applied Mathematics and Computation, 220, 455–462.

16. Al-Amin, M., Islam, M. N., İlhan, O. A., Akbar, M. A., Soybaş, D. (2022). Solitary wave solutions to the modified Zakharov-Kuznetsov and the (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff models in mathematical physics. Journal of Mathematics, 2022, 5224289.

17. Roshid, H. O. (2019). Multi-soliton of the (2+ 1)-dimensional Calogero-Bogoyavlenskii-Schiff equation and KdV equation. Computational Methods for Differential Equations, 7(1), 86-95.

18. Cinar, M., Secer, A., Bayram, M. (2022). Analytical solutions of (2+1)-dimensional Calogero-Bogoyavlenskii-Schiff equation in fluid mechanics/plasma physics using the New Kudryashov method. Physica Scripta, 97(9), 094002.

19. Singh, D., Yadav, S., Arora, R. (2022). A (2+1)-dimensional modified dispersive water-wave (MDWW) system: Lie symmetry analysis, optimal system and invariant solutions. Communications in Non-linear Science and Numerical Simulation, 115, 106786.

20. Chauhan, A., Arora, R., Tomar, A. (2020). Lie symmetry analysis and traveling wave solutions of equal width wave equation. Proyecciones (Antofagasta), 39(1), 179-198.

21. Wambura, K. O., Okoya, M. O., Aminer, T. J. O. (2019). Lie symmetry analysis and the optimal system of non-linear fourth order evolution equation. International Journal of Scientific Engineering and Applied Science, 5(3), 1-6.

22. Yadav, S., Chauhan, A., Arora, R. (2021). Invariance analysis, optimal system and conservation laws of (2+1)-dimensional non-linear Vakhnenko equation. Pramana, 95, 1-13.

23. Kumar, S., Dhiman, S. K. (2022). Lie symmetry analysis, optimal system, exact solutions and dynamics of solitons of a (3+1)-dimensional generalised BKP–Boussinesq equation. Pramana, 96(1), 31.

24. Chauhan, A., Arora, R. (2020). Some exact solutions of (1+1)-dimensional Kaup-system and seventh-order Kawahara equation. Malaya Journal of Matematik (MJM), 8(1), 151-158.

25. Amata, L., Oliveri, F., Sgroi, E. (2024). Optimal systems of Lie subalgebras: A computational approach. Journal of Geometry and Physics, 204, 105290.

26. Bluman, G. W., Kumei, S. (1989). Symmetries and Differential Equations. Springer New York, 412, 1-150.

27. Whitham, G. B. (1974). Linear and Non-linear Waves. John Wiley and Sons, New York, 850, 533-576.